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NON-UNIFORM TIMOSHENKO BEAMS WITH TIME-DEPENDENT ELASTIC BOUNDARY CONDITIONS

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A solution procedure for studying the dynamic responses of a non-uniform Timoshenko beam with general time-dependent boundary conditions is developed by generalizing the method of Mindlin–Goodman and utilizing the exact solutions of non-uniform Timoshenko beam vibration given by Lee and Lin. A general change of dependent variable with shifting functions is introduced and the physical meanings of these shifting functions are further explored. The orthogonality condition for the eigenfunctions of a non-uniform Timoshenko beam with elastic boundary conditions is also derived. Several limiting cases and their corresponding procedures are revealed. Finally, the influence of the spring constant on the steady response of a beam subjected to a harmonic base excitation is investigated.

1. INTRODUCTION

In many structural fields such as the aircraft and space structures under ground vibration testing, a building, a bridge and a highway construction subjected to an earthquake, the vibration of a structure can be mathematically modelled as the transverse vibration of a beam subjected to time-dependent support motion.

The vibrational problem has been studied by many investigators via Bernoulli–Euler and Timoshenko beam theory, respectively. The vibration of a uniform Bernoulli–Euler beam with classical time dependent boundary conditions can be solved by using the method of Laplace transform [1, 2] and the method of Mindlin–Goodman [3–6]. In the Mindlin–Goodman method, a change of dependent variable together with four shifting polynomial functions of the fifth order is introduced. In general, by properly selecting these shifting polynomial functions, the original system will be transformed to be a system composed of a non-homogeneous governing differential equation and four homogeneous

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boundary conditions. Consequently, the method of separation of variables was used to solve the problem. Edstrom [4] pointed out that if a properly chosen change of dependent variable is made, an original system composed of a homogeneous governing differential equation and four non-homogeneous boundary conditions will be transformed to be a system composed of a homogeneous governing differential equation and four homogeneous boundary conditions. However, the method is not suitable for the system subjected to transverse forces. Recently, the dynamic analysis of non-uniform Bernoulli–Euler beams with time dependent elastic boundary conditions was given by Lee and Lin [7]. They generalized the method of Mindlin–Goodman and introduced four shifting polynomial functions of the third degree, instead of the polynomial functions of the fifth degree given by Mindlin and Goodman [3]. The physical meanings of those shifting functions were provided. Nevertheless, the physical meanings of those shifting functions are suitable for uniform beam system and are not consistent with non-uniform beam systems.

The vibration of uniform Timoshenko beams with classical time dependent boundary conditions was studied by Herrmann [8, 9] and Berry and Naghdi [10] by using the method of Mindlin and Goodman [3]. However, in their studies polynomial functions were chosen as shifting functions and were required to satisfy the boundary conditions only. Hence, the shifting functions did not have any physical meanings, and no general form was given for various kind of boundary conditions. In addition, the method will lead to considerable difficulties when taking the limiting study from the Timoshenko beam theory to the Bernoulli–Euler beam theory.

In this paper, the previous study made by Lee and Lin [7] is extended and the method of Mindlin–Goodman further generalized to develop a solution procedure for studying the vibrations of a non-uniform Timoshenko beam with general time-dependent boundary conditions. First, the time-dependent elastic boundary conditions are formulated. A general change of dependent variable with shifting functions is introduced. Shifting functions are selected which are justified for the non-uniform beam system and different from those given by Herrmann [9] and Lee and Lin [7]. The physical meanings of these shifting polynomial functions are explored. In addition, the work done by Dolph [11] who derived the orthogonality condition for the eigenfunctions of a uniform Timoshenko beam with classical boundary conditions is extended, and the orthogonality condition for the eigenfunctions of a non-uniform Timoshenko beam with elastic boundary conditions is derived. The limiting study from the Timoshenko beam theory to the Bernoulli–Euler beam theory is also revealed. With the present approach, the difficulties, in the Herrmann [9] and Berry and Nagdhi [10] approach, in taking the limiting study are overcome. Finally, several limiting cases are examined.

2. NON-UNIFORM TIMOSHENKO BEAMS

Consider a non-uniform Timoshenko beam with time-dependent elastic boundary conditions, as shown in Figure 1. The following dimensionless quantities are introduced,

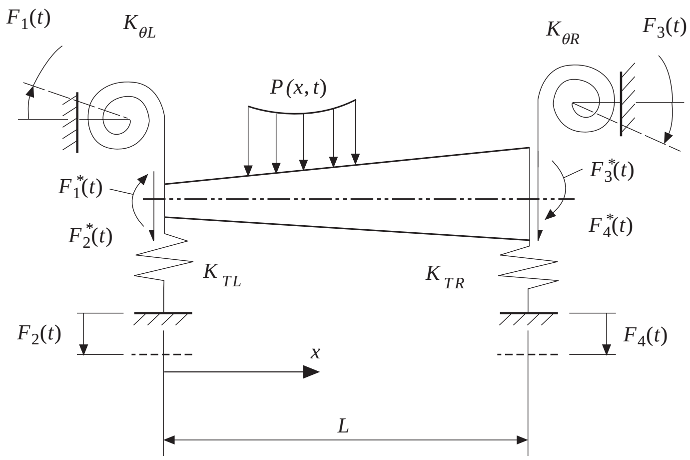

Figure 1. Geometry and co-ordinate system of a non-uniform Timoshenko beam with time dependent elastic boundary conditions, subjected to transverse force and distributed moment.

$$b(\xi) = E(x)I(x)/E(0)I(0), \qquad f_1(\tau) = F_1(t), \qquad f_2(\tau) = F_2(t)/L, \qquad f_3(\tau) = F_3(t),$$
$$f_4(\tau) = F_4(t)/L, \qquad f_1^*(\tau) = F_1^*(t)L/E(0)I(0),$$
$$f_2^*(\tau) = F_2^*(t)L^2/E(0)I(0), \qquad f_3^*(\tau) = F_3^*(t)L/E(0)I(0),$$
$$f_4^*(\tau) = F_4^*(t)L^2/E(0)I(0),$$
$$m(\xi) = \rho(x)A(x)/\rho(0)A(0), \qquad \bar{m}(\xi,\tau) = M(x,t)L^2/E(0)I(0),$$
$$p(\xi,\tau) = P(x,t)L^3/E(0)I(0), \qquad q(\xi) = J(x)/J(0),$$
$$s(\xi) = \kappa(x)G(x)A(x)/\kappa(0)G(0)A(0), \qquad w(\xi,\tau) = W(x,t)/L,$$
$$\beta_1 = K_{\theta L}L/E(0)I(0), \qquad \beta_2 = K_{TL}L^3/E(0)I(0), \qquad \beta_3 = K_{\theta R}L/E(0)I(0),$$
$$\beta_4 = K_{TR}L^3/E(0)I(0), \qquad \tau = (t/L^2)\sqrt{E(0)I(0)/\rho(0)A(0)},$$
$$\xi = x/L, \qquad \eta = J(0)/\rho(0)A(0)L^2, \qquad \mu = E(0)I(0)/\kappa(0)G(0)A(0)L^2. \tag{1}$$

The two coupled dimensionless governing differential equations of the system are

$$-\partial/\partial\xi[(s(\xi)/\mu)(\partial w/\partial\xi - \Psi)] + m(\xi)\,\partial^2 w/\partial\tau^2 = p(\xi,\tau), \tag{2}$$
$$(\partial/\partial\xi)[b(\xi)\,\partial\Psi/\partial\xi] + (s(\xi)/\mu)(\partial w/\partial\xi - \Psi) - \eta q(\xi)\,\partial^2\Psi/\partial\tau^2 = -\bar{m}(\xi,\tau). \tag{3}$$

At $\xi = 0$:

$$\partial\Psi/\partial\xi - \beta_1\Psi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{4}$$
$$-(1/\mu)(\partial w/\partial\xi - \Psi) + \beta_2 w = \beta_2 f_2(\tau) + f_2^*(\tau); \tag{5}$$

at $\xi = 1$:

$$b\,\partial\Psi/\partial\xi - \beta_3\Psi = \beta_3 f_3(\tau) + f_3^*(\tau), \qquad (s/\mu)(\partial w/\partial\xi - \Psi) + \beta_4 w = \beta_4 f_4(\tau) + f_4^*(\tau), \tag{6, 7}$$

and the associated dimensionless initial conditions of the motion are

$$w(\xi, 0) = w_0(\xi), \qquad \Psi(\xi, 0) = \Psi_0(\xi), \qquad \partial w(\xi, 0)/\partial\tau = \dot{w}_0(\xi), \qquad (8\text{–}10)$$

$$\partial\Psi(\xi, 0)/\partial\tau = \dot{\Psi}_0(\xi), \qquad (11)$$

where $w(x, t)$ is the flexural displacement, Ψ is the angle of rotation due to bending, $G(x)$ is the shear modulus, $E(x)$ is the Young's modulus and κ is the shear correction factor, x is the co-ordinate along the beam, t is time and L is the length of the beam. $I(x)$, $J(x)$ and $A(x)$ denote the area moment of inertia, the mass moment of inertia per unit length about the neutral axis and the cross sectional area, respectively. $\rho(x)$ is the mass density per unit volume and $P(x, t)$ and $M(x, t)$ are the applied transverse force and distributed moment per unit length, respectively. $F_1(t)$, $F_2(t)$, $F_1^*(t)$ and $F_2^*(t)$ and $F_3(t)$, $F_4(t)$, $F_3^*(t)$ and $F_4^*(t)$ are the slope, the displacement, the external moment and the shear force excitations at the left end and the right end of the beam, respectively. K_{TL} and $K_{\theta L}$ and K_{TR} and $K_{\theta R}$ are the translational spring constants and the rotational spring constants at the left end and the right end of the beam, respectively. $w_0(\xi)$, $\psi_0(\xi)$, $\dot{w}_0(\xi)$, and $\dot{\Psi}_0(\xi)$ are four prescribed dimensionless initial functions.

When the dimensionless translational spring constant is infinity or zero, the time dependent displacement or the time dependent shear force is prescribed. If the dimensionless rotational spring constant is infinity or zero, then the time dependent angle of rotation due to bending or the time dependent moment is prescribed.

3. SOLUTION METHOD

3.1. CHANGE OF VARIABLE

To find the solution for these differential equations with variable coefficients and non-homogeneous elastic boundary conditions, one generalizes the method given by Mindlin and Goodman [3] and Herrmann [9] by taking

$$w(\xi, \tau) = v(\xi, \tau) + \sum_{i=1}^{4} \bar{f}_i(\tau) g_i(\xi), \qquad \Psi(\xi, \tau) = \varphi(\xi, \tau) + \sum_{i=1}^{4} \bar{f}_i(\tau) \bar{g}_i(\xi), \qquad (12, 13)$$

where

$$\bar{f}_i(\tau) = (\beta_i/[1 + \beta_i]) f_i(\tau) + (1/[1 + \beta_1]) f_i^*(\tau), \qquad i = 1, 2, 3, 4, \qquad (14)$$

and the shifting functions $g_i(\xi)$ and $\bar{g}_i(\xi)$, $i = 1, 2, 3, 4$, are chosen to satisfy the following two differential equations

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{s(\xi)}{\mu}\left(\frac{\mathrm{d}g_i}{\mathrm{d}\xi} - \bar{g}_i\right)\right] = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\left[b(\xi)\frac{\mathrm{d}\bar{g}_i}{\mathrm{d}\xi}\right] + \frac{s(\xi)}{\mu}\left(\frac{\mathrm{d}g_i}{\mathrm{d}\xi} - \bar{g}_i\right) = 0, \quad i = 1, 2, 3, 4 \qquad (15, 16)$$

and the boundary conditions:

$$\frac{\beta_1}{1+\beta_1}\bar{g}_i - \frac{1}{1+\beta_1}\frac{\mathrm{d}\bar{g}_i}{\mathrm{d}\xi}\bigg|_{\xi=0} = \delta_{i1}, \qquad \frac{\beta_2}{1+\beta_2}g_i + \frac{1}{1+\beta_2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[b(\xi)\frac{\mathrm{d}\bar{g}_i}{\mathrm{d}\xi}\right]\bigg|_{\xi=0} = \delta_{i2}, \tag{17, 18}$$

$$\frac{\beta_3}{1+\beta_3}\bar{g}_i + \frac{b}{1+\beta_3}\frac{\mathrm{d}\bar{g}_i}{\mathrm{d}\xi}\bigg|_{\xi=1} = \delta_{i3}, \qquad \frac{\beta_4}{1+\beta_4}g_i - \frac{1}{1+\beta_4}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[b(\xi)\frac{\mathrm{d}\bar{g}_i}{\mathrm{d}\xi}\right]\bigg|_{\xi=1} = \delta_{i4}, \tag{19, 20}$$

where δ_{ij} is a Kronecker symbol. After substituting equations (12–20) into equations (2–11), one has the following differential equations in terms of $v(\xi, \tau)$ and $\varphi(\xi, \tau)$,

$$-(\partial/\partial\xi)[(s(\xi)/\mu)(\partial v/\partial\xi - \varphi)] + m(\xi)\,\partial^2 v/\partial\tau^2 = p_1(\xi, \tau), \tag{21}$$

$$(\partial/\partial\xi)[b(\xi)\,\partial\varphi/\partial\xi] + s(\xi)/\mu(\partial v/\partial\xi - \varphi) - \eta q(\xi)\,\partial^2\varphi/\partial\tau^2 = p_2(\xi, \tau), \tag{22}$$

where

$$p_1(\xi, \tau) = p(\xi, \tau) - \sum_{i=1}^{4} m(\xi) g_i(\xi)\frac{\mathrm{d}^2\bar{f}_i}{\mathrm{d}\tau^2}, \tag{23}$$

$$p_2(\xi, \tau) = -\bar{m}(\xi, \tau) + \sum_{i=1}^{4} \eta q(\xi)\bar{g}_i(\xi)\frac{\mathrm{d}^2\bar{f}_i}{\mathrm{d}\tau^2}, \tag{24}$$

and the associated homogeneous boundary conditions:

at $\xi = 0$:

$$\partial\varphi/\partial\xi - \beta_1\varphi = 0, \qquad (-1/\mu)(\partial v/\partial\xi - \varphi) + \beta_2 v = 0; \tag{25, 26}$$

and at $\xi = 1$:

$$b\,\partial\varphi/\partial\xi + \beta_3\varphi = 0, \qquad (s/\mu)(\partial v/\partial\xi - \varphi) + \beta_4 v = 0. \tag{27, 28}$$

The transformed initial conditions (8–11) become

$$v(\xi, 0) = w_0(\xi) - \sum_{i=1}^{4}\bar{f}_i(0) g_i(\xi), \qquad \varphi(\xi, 0) = \Psi_0(\xi) - \sum_{i=1}^{4}\bar{f}_i(0)\bar{g}_i(\xi), \tag{29, 30}$$

$$\frac{\partial v(\xi, 0)}{\partial\tau} = \dot{w}_0(\xi) - \sum_{i=1}^{4}\frac{\mathrm{d}\bar{f}_i(0)}{\mathrm{d}\tau} g_i(\xi), \qquad \frac{\partial\varphi(\xi, 0)}{\partial\tau} = \dot{\Psi}_0(\xi) - \sum_{i=1}^{4}\frac{\mathrm{d}\bar{f}_i(0)}{\mathrm{d}\tau}\bar{g}_i(\xi). \tag{31, 32}$$

It should be mentioned that if the beam is uniform and the elastic spring constants β_i are set to be infinity or zero, then equations (2–14) and (17–20) reduce to those given by Herrmann [9].

3.2. SHIFTING FUNCTIONS AND THEIR PHYSICAL MEANINGS

The shifting functions $g_i(\xi)$ and $\bar{g}_i(\xi)$ which satisfy equations (15–16) can be written as

$$\bar{g}_i(\xi) = \int_0^{\xi} \frac{2\alpha_{i,2} + 6\alpha_{i,3}\zeta}{b(\zeta)}\, \mathrm{d}\zeta + \alpha_{i,1} + 6\mu\alpha_{i,3}, \tag{33}$$

$$g_i(\xi) = \int_0^{\xi} [\bar{g}_i(\zeta) - 6\mu\alpha_{i,3}/s(\zeta)]\, \mathrm{d}\zeta + \alpha_{i,0}, \qquad i = 1, 2, 3, 4, \tag{34}$$

where $\alpha_{i,0}$, $\alpha_{i,1}$, $\alpha_{i,2}$ and $\alpha_{i,3}$ are constants to be specified from the boundary conditions, equations (17–20). These constants for the general and limiting cases are derived and listed in the Appendix.

The shifting functions $g_i(\xi)$ and $\bar{g}_i(\xi)$, $i = 1, 2, 3, 4$ can be interpreted as the static deflection and the angle of rotation due to bending of a generally elastically restrained non-uniform Timoshenko beam subjected only to a unit moment and a unit slope of the base at the left end, a unit shear force and a unit displacement of the base at the left end, a unit moment and a unit slope of the base at the right end and a unit shear force and a unit displacement of the base at the right end of the beam, respectively.

3.3. SHIFTING FUNCTIONS OF UNIFORM BEAMS

When the beam is uniform, then $b(\xi) = 1$ and $s(\xi) = 1$. The shifting functions (33–34) become

$$g_i(\xi) = \alpha_{i,0} + \alpha_{i,1}\xi + \alpha_{i,2}\xi^2 + \alpha_{i,3}\xi^3, \tag{35}$$

$$\bar{g}_i(\xi) = \alpha_{i,1} + 6\mu\alpha_{i,3} + 2\alpha_{i,2}\xi + 3\alpha_{i,3}\xi^2, \qquad i = 1, 2, 3, 4. \tag{36}$$

It should be mentioned that the shifting functions $g_i(\xi)$ and $\bar{g}_i(\xi)$ in Herrmann's approach are two individual polynomials and are required to satisfy the typical elastic boundary conditions which are the limiting cases of the present study (17–20). Those shifting functions do not have any physical meanings and will lead to considerable difficulties when taking the limiting study from Timoshenko beam theory to Bernoulli–Euler beam theory. In the present approach they are required to satisfy both the differential equations (15–16) and the boundary conditions (17–20). The shifting functions have only four constants to be determined and are simpler than those given by Herrmann [9]. They can also be used to cover the very general case. In addition, these shifting functions take physical meanings.

3.4. ORTHOGONALITY CONDITION

The solution for equations (21–32), $v(\xi, \tau)$ and $\varphi(\xi, \tau)$ can be obtained by using the method of eigenfunction expansion. The eigenfunctions are specified by the associated homogeneous governing differential equations and homogenous boundary conditions.

To derive the orthogonality condition of the eigenfunctions of the system, one lets ω_n^2 be the nth eigenvalue or the square of the nth natural frequency and $[v_n(\xi)\varphi_n(\xi)]^T$ be the nth eigenfunction of the system, where the superscript T is the symbol of the transpose of a matrix. The governing characteristic differential equation can be expressed as

$$\{[\mathbf{L}] + \omega_n^2[\mathbf{M}]\}\begin{bmatrix} v_n(\xi) \\ \varphi_n(\xi) \end{bmatrix} = 0, \tag{37}$$

where the differential operators $[\mathbf{L}]$ and $[\mathbf{M}]$ are

$$[\mathbf{L}] = \begin{bmatrix} (\mathrm{d}/\mathrm{d}\xi)[(s(\xi)/\mu)(\mathrm{d}/\mathrm{d}\xi)] & -(\mathrm{d}/\mathrm{d}\xi)[s(\xi)/\mu] \\ (s(\xi)/\mu)\,\mathrm{d}/\mathrm{d}\xi & (\mathrm{d}/\mathrm{d}\xi)[b(\xi)\,\mathrm{d}/\mathrm{d}\xi] - s(\xi)/\mu \end{bmatrix}, \tag{38}$$

and

$$[\mathbf{M}] = \begin{bmatrix} m(\xi) & 0 \\ 0 & \eta q(\xi) \end{bmatrix}, \tag{39}$$

respectively. The eigenfunctions satisfy the boundary conditions (25–28). It can be observed that equations (37) and the associated boundary conditions take the meaning of the free vibration of an elastically restrained non-uniform Timoshenko beam. The eigenfunctions and the eigenvalues can be obtained by using the method proposed by Lee and Lin [12]. They decoupled the coupled differential equations into two complete fourth order differential equations in the flexural displacement and in the angle of rotation due to bending, respectively. It was shown that if the geometric and the material properties of the beam can be expressed in polynomial forms, then the exact solutions of the system can be obtained.

Taking the inner product, one can easily show that

$$\int_0^1 [v_j(\xi)\varphi_j(\xi)][\mathbf{M}]\begin{bmatrix} v_n(\xi) \\ \varphi_n(\xi) \end{bmatrix}\mathrm{d}\xi = \int_0^1 [v_n(\xi)\varphi_n(\xi)][\mathbf{M}]\begin{bmatrix} v_j(\xi) \\ \varphi_j(\xi) \end{bmatrix}\mathrm{d}\xi, \tag{40}$$

$$\begin{aligned} \int_0^1 [v_j(\xi)\varphi_j(\xi)][\mathbf{L}]\begin{bmatrix} v_n(\xi) \\ \varphi_n(\xi) \end{bmatrix}\mathrm{d}\xi &= \int_0^1 [v_n(\xi)\varphi_n(\xi)][\mathbf{L}]\begin{bmatrix} v_j(\xi) \\ \varphi_j(\xi) \end{bmatrix}\mathrm{d}\xi \\ &\quad + \{v_j[(s/\mu)(\mathrm{d}v_n/\mathrm{d}\xi - \varphi_n)] - v_n[(s/\mu)(\mathrm{d}v_j/\mathrm{d}\xi - \varphi_j)]\}|_0^1 \\ &\quad + b\{\varphi_j\,\mathrm{d}\varphi_n/\mathrm{d}\xi - \varphi_n\,\mathrm{d}\varphi_j/\mathrm{d}\xi\}|_0^1. \end{aligned} \tag{41}$$

The two boundary terms in equation (41) vanish because of the boundary conditions (25–28). Thus the self-adjointness of the system is proved. Consequently, the following orthogonality condition is proved

$$\int_0^1 [v_j(\xi)\varphi_j(\xi)]\begin{bmatrix} m(\xi) & 0 \\ 0 & \eta q(\xi)\end{bmatrix}\begin{bmatrix} v_n(\xi) \\ \varphi_n(\xi)\end{bmatrix} \mathrm{d}\xi = \left\{\begin{matrix} 0, & j \neq n, \\ \epsilon_n, & j = n, \end{matrix}\right\} \tag{42}$$

where ϵ_n is a real number. For a uniform beam, $m(\xi) = 1$ and $\eta q(\xi) = \eta$, and the orthogonality condition (42) becomes the same as that given by Dolph [11].

3.5. THE MODE SUPERPOSITION

The solution $v(\xi, \tau)$ and $\varphi(\xi, \tau)$ specified by equations (21–32) can be expressed in the following eigenfunction expansion form

$$\begin{bmatrix} v(\xi, \tau) \\ \varphi(\xi, \tau)\end{bmatrix} = \sum_{n=0}^{\infty} T_n(\tau)\begin{bmatrix} v_n(\xi) \\ \varphi_n(\xi)\end{bmatrix}. \tag{43}$$

Substituting it back to the governing equations (21–22) and the initial conditions (29–32), multiplying by $[v_n(\xi)\varphi_n(\xi)]$ and integrating in accordance with the orthogonality condition (42), one obtains

$$\frac{\mathrm{d}^2 T_n}{\mathrm{d}\tau^2} + \omega_n^2 T_n = \frac{-1}{\epsilon_n}\int_0^1 [v_n(\xi)p_1(\xi, \tau) + \varphi_n(\xi)p_2(\xi, \tau)]\, \mathrm{d}\xi. \tag{44}$$

The corresponding initial conditions are

$$T_n(0) = \frac{1}{\epsilon_n}\int_0^1 [m(\xi)v_n(\xi)v(\xi, 0) + \eta q(\xi)\varphi_n(\xi)\varphi(\xi, 0)]\, \mathrm{d}\xi, \tag{45}$$

$$\frac{\mathrm{d}T_n(0)}{\mathrm{d}\tau} = \frac{1}{\epsilon_n}\int_0^1 [m(\xi)v_n(\xi)\frac{\partial v(\xi, 0)}{\partial \tau} + \eta q(\xi)\varphi_n(\xi)\frac{\partial \varphi(\xi, 0)}{\partial \tau}]\, \mathrm{d}\xi. \tag{46}$$

The solution is

$$T_n(\tau) = T_n(0)\cos\omega_n\tau + \frac{1}{\omega_n}\frac{\mathrm{d}T_n(0)}{\mathrm{d}\tau}\sin\omega_n\tau + \frac{1}{\omega_n}\int_0^\tau p_n^*(\zeta)\sin\omega_n(\tau - \zeta)\, \mathrm{d}\zeta, \tag{47}$$

where $p_n^*(\tau)$ is the forced term in equation (44). After substituting equation (47) back to equation (43), finally, the general forced response of the beam with time dependent boundary conditions is obtained by substituting the shifting functions (33–34) and the solution (43) into equations (12–14). This completes the solution of the system.

It should be mentioned that the static deflection of a non-uniform Timoshenko beam with non-homogenous boundary conditions can also be obtained through the dynamic solution by eliminating the time dependent parameters and terms in the solution procedures.

4. BERNOULLI–EULER BEAMS

For a Bernoulli–Euler beam, the shear deformation and the rotatory inertia are not considered, i.e., η and μ approach zero. Equations (3) and (22) are reduced to the following two representations for the shear force and the transformed shear force

$$\frac{s(\xi)}{\mu}\left(\frac{\partial w}{\partial \xi}-\Psi\right)=-\frac{\partial}{\partial \xi}\left[b(\xi)\frac{\partial \Psi}{\partial \xi}\right]-\bar{m}(\xi,\tau), \tag{48}$$

$$\frac{s(\xi)}{\mu}\left(\frac{\partial v}{\partial \xi}-\varphi\right)=-\frac{\partial}{\partial \xi}\left[b(\xi)\frac{\partial \varphi}{\partial \xi}\right]-\bar{m}(\xi,\tau). \tag{49}$$

Without considering the applied distributed moment and substituting equation (48) into equation (2) and the boundary conditions (4–7), gives the governing differential equation and the boundary conditions of a non-uniform Bernoulli–Euler beam with time dependent elastic boundary conditions. They are exactly the same as those given by Lee and Lin [7].

From equation (34), it is also observed that the first derivative of the shifting function $g_i(\xi)$ is equal to $\bar{g}_i(\xi)$

$$\mathrm{d}g_i(\xi)/\mathrm{d}\xi=\bar{g}_i(\xi). \tag{50}$$

From equations (12–13) and (50) and the Bernoulli–Euler beam theory (in which the angle of rotation due to bending is equal to the slope of the beam), one has

$$\partial v(\xi,\tau)/\partial \xi=\varphi(\xi,\tau), \tag{51}$$

which can be alternatively obtained from equation (49).

Applying the relations (48–51) to equations (21–32), the governing differential equation for the transformed variable $v(\xi,\tau)$ is reduced to

$$(\partial^2/\partial\xi^2)[b(\xi)\,\partial^2 v/\partial\xi^2]+m(\xi)(\partial^2 v/\partial\tau^2)=p_1(\xi,\tau)-\partial\bar{m}(\xi,\tau)/\partial\xi, \tag{52}$$

the associated boundary conditions are

at $\xi=0$:

$$\partial^2 v/\partial\xi^2-\beta_1\,\partial v/\partial\xi=0,\qquad (\partial/\partial\xi)(b(\xi)\,\partial^2 v/\partial\xi^2)+\beta_2 v=0; \tag{53, 54}$$

at $\xi=1$:

$$b\,\partial^2 v/\partial\xi^2+\beta_3\,\partial v/\partial\xi=0,\qquad -(\partial/\partial\xi)(b(\xi)\,\partial^2 v/\partial\xi^2)+\beta_4 v=0, \tag{55, 56}$$

and the associated initial conditions are

$$v(\xi,0)=w_0(\xi)-\sum_{i=1}^{4}\bar{f}_i(0)g_i(\xi),\qquad \frac{\partial v(\xi,0)}{\partial\tau}=\dot{w}_0(\xi)-\sum_{i=1}^{4}\frac{\mathrm{d}\bar{f}_i(0)}{\mathrm{d}\tau}g_i(\xi), \tag{57, 58}$$

The shifting functions $g_i(\xi)$, $i = 1, 2, 3, 4$, can be obtained from those of Timoshenko beams by letting $\mu = 0$ and the solution v can be obtained by the solution method developed by Lee and Kuo [13] or the method of eigenfunction expansion reduced from equations (43–47).

5. VERIFICATION AND EXAMPLES

To verify the previous analysis, two examples are illustrated.

Example 1: Consider the vibration of a simply supported uniform beam subjected to a step moment excitation at the left end of the beam with the following initial conditions

$$w(\xi, 0) = \Psi(\xi, 0) = \partial w(\xi, 0)/\partial\tau = \partial\Psi(\xi, 0)/\partial\tau = 0. \tag{59}$$

The end excitation functions are

$$f_1 = f_2 = f_3 = f_4 = f_2^* = f_3^* = f_4^* = 0, \qquad \bar{f}_1 = f_1^* = \begin{Bmatrix} 0, & \tau < 0, \\ -\alpha, & \tau \geqslant 0, \end{Bmatrix} \tag{60}$$

where α is a positive constant. The associated shifting functions $g_i(\xi)$ and $\bar{g}_i(\xi)$, $i = 1, 2, 3, 4$, are those listed on case 2 of the Appendix. The governing characteristic equation (37) now is

$$\left\{\begin{bmatrix} (1/\mu)\,\mathrm{d}^2/\mathrm{d}\xi^2 & (-1/\mu)\,\mathrm{d}/\mathrm{d}\xi \\ (1/\mu)\,\mathrm{d}/\mathrm{d}\xi & \mathrm{d}^2/\mathrm{d}\xi^2 - 1/\mu \end{bmatrix} + \omega_n^2 \begin{bmatrix} 1 & 0 \\ 0 & \eta \end{bmatrix}\right\} \begin{bmatrix} v_n(\xi) \\ \varphi_n(\xi) \end{bmatrix} = 0, \tag{61}$$

and the associated boundary conditions are

at $\xi = 0$:

$$\partial\varphi_n/\partial\xi = 0, \qquad v_n = 0, \tag{62, 63}$$

at $\xi = 1$:

$$\partial\varphi_n/\partial\xi = 0, \qquad v_n = 0. \tag{64, 65}$$

Following the method developed by Lee and Lin [12], the eigenvalues can be obtained

$$\omega_{n,1} = ([n^2\pi^2(\eta + \mu) + 1] + \sqrt{[n^2\pi^2(\eta + \mu) + 1] - 4\eta\mu n^4\pi^4})/2\eta\mu, \tag{66a}$$

$$\omega_{n,2} = ([n^2\pi^2(\eta + \mu) + 1] - \sqrt{[n^2\pi^2(\eta + \mu) + 1] - 4\eta\mu n^4\pi^4})/2\eta\mu, \quad n = 0, 1, 2, \ldots \tag{66b}$$

and the associated eigenfunctions are

$$\varphi_n(\xi) = -\cos n\pi\xi, \qquad v_{n,i}(\xi) = \frac{n\pi}{\mu\omega_{n,i}^2 - n^2\pi^2}\sin n\pi\xi,$$
$$i = 1, 2 \qquad n = 0, 1, 2, \ldots \tag{67a, b}$$

When the external load $p(\xi, \tau)$ and the external moment $\bar{m}(\xi, \tau)$ are assumed to be zero, the displacement and the angle of rotation due to bending can be obtained, respectively

$$\Psi = -\alpha\left\{\left(\tfrac{1}{3} + \mu - \xi + \tfrac{1}{2}\xi^2\right) - \mu \cos \omega_{0,1}\tau \right.$$

$$\left. + 2\sum_{n=1}^{\infty} \cos n\pi\xi \sum_{i=1}^{2} \frac{(\mu\omega_{n,i}^2 - n^2\pi^2)[1 - \eta(\mu\omega_{n,i}^2 - n^2\pi^2)]}{n^2\pi^2[n^2\pi^2 + \eta(\mu\omega_{n,i}^2 - n^2\pi^2)^2]} \cos \omega_{n,i}\tau \right\}, \qquad (68a)$$

$$w(\xi, \tau) = -\alpha\left\{\tfrac{1}{3}\xi - \tfrac{1}{2}\xi^2 + \tfrac{1}{6}\xi^3 \right.$$

$$\left. - 2\sum_{n=1}^{\infty} \sin n\pi\xi \sum_{i=1}^{2} \frac{1 - \eta(\mu\omega_{n,i}^2 - n^2\pi^2)}{n\pi[n^2\pi^2 + \eta(\mu\omega_{n,i}^2 - n^2\pi^2)^2]} \cos \omega_{n,i}\tau \right\}. \qquad (68b)$$

These solutions are the same as those given by Berry and Naghdi [10].

Example 2: Consider the vibration of a clamped–elastically restrained non-uniform beam with constant width and linearly varying depth, subjected to a harmonic base excitation at the right end of the beam. The dimensionless material properties, applied transverse force and moment are

$$m(\xi) = s(\xi) = 1 + \lambda\xi, \qquad b(\xi) = q(\xi) = (1 + \lambda\xi)^3, \qquad \bar{m}(\xi, \tau) = p(\xi, \tau) = 0. \qquad (69)$$

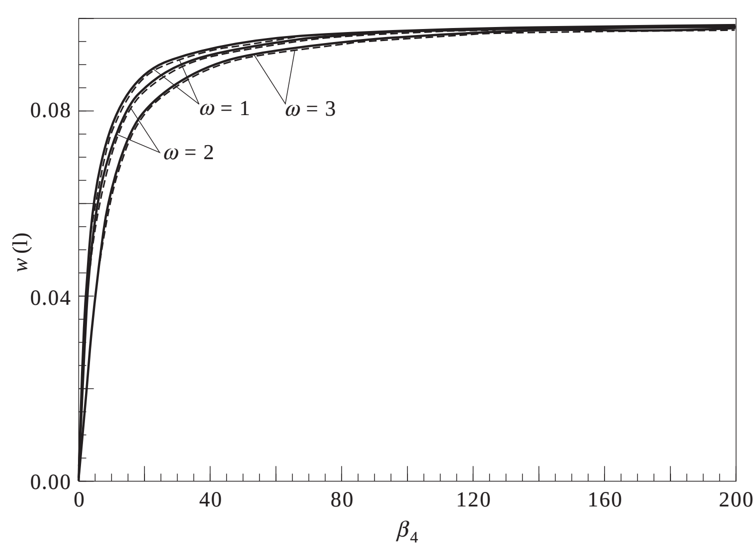

Figure 2. The influence of the translational spring constant β_4 on the amplitude of steady response at the right end of the non-uniform beam subjected to harmonic base excitation (β_1, $\beta_2 \to \infty$, $\beta_3 = 0$, $b = q = (1 - 0{\cdot}1\xi)^3$, $m = s = (1 - 0{\cdot}1\xi)$, $\gamma_0 = 0{\cdot}1$): Key: ——, $\eta = 0{\cdot}01$, $\mu = 0{\cdot}0312$; ----, $\eta = \mu = 0$.

The excitation functions are

$$f_1 = f_2 = f_3 = f_1^* = f_2^* = f_3^* = f_4^* = \bar{f}_1 = \bar{f}_2 = \bar{f}_3 = 0, \qquad \bar{f}_4 = f_4 = \gamma_0 \sin(\omega\tau). \tag{70}$$

The associated shifting functions g_i, $i = 1, 2, 3, 4$ are those listed in case 1 of the Appendix.

In Figure 2, the influence of the translational spring constant β_4 on the amplitude of steady response at the right end of the non-uniform beam subjected to harmonic base excitation is illustrated. When β_4 is zero, there is no influence of the excitation on the vibration of the beam. When β_4 is increased, the amplitude is evidently increased. When β_4 approaches infinity, the amplitude of the tip of the beam is the same as that of the excitation. Given the translational spring constant β_4, the higher the frequency of the excitation, the lower the amplitude of the tip of the beam. There is almost no difference of the above response between a Timoshenko beam and a Bernoulli–Euler beam.

6. CONCLUSION

A systematic solution procedure for studying the dynamic responses of a non-uniform Timoshenko beam with general time-dependent elastic boundary conditions has been developed by generalizing the method of Mindlin–Goodman and utilizing the exact solutions of non-uniform Timoshenko beam vibration given by Lee and Lin. A general change of dependent variable with shifting functions is introduced and the physical meanings of these shifting functions are explored. The dynamic responses of a non-uniform Bernoulli–Euler beam can be obtained easily just by taking a suitable limiting procedure. Meanwhile, the physical meanings of the reduced corresponding shifting functions do exist. The self-adjointness of a Timoshenko beam system is proved. The orthogonality condition for the eigenfunctions of a non-uniform Timoshenko beam with elastic boundary conditions is also derived. The static deflection of a non-uniform Timoshenko beam with non-homogenous boundary conditions can also be obtained from the dynamic solution by eliminating the time dependent parameters and terms in the solution procedures. There is almost no difference of the steady response between a Timoshenko beam and a Bernoulli–Euler beam subjected to a harmonic base excitation.

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APPENDIX: SHIFTING FUNCTIONS

CASE 1: GENERAL CASE

In this case, β_1, β_2, β_3 and β_4 are constant, the coefficients of the shifting functions are

$$\alpha_{1,0} = 6\gamma_{22}(\gamma_{31}A_3 - \gamma_{41}A_1)/H, \qquad \alpha_{1,1} = \{\gamma_{21}(A_2A_3 - A_1A_4) + 6\gamma_{22}\gamma_{41}A_1]\}/H,$$

$$\alpha_{1,2} = [\gamma_{21}(\gamma_{31}A_4 - \gamma_{41}A_2) - 6\gamma_{22}\gamma_{31}\gamma_{41}]/H, \qquad \alpha_{1,3} = \gamma_{21}(\gamma_{41}A_1 - \gamma_{31}A_3)/H;$$

$$\alpha_{2,0} = [\gamma_{11}(A_2A_3 - A_1A_4) + 2\gamma_{12}(\gamma_{31}A_4 - \gamma_{41}A_2) + 6\mu\gamma_{11}(\gamma_{31}A_3 - \gamma_{41}A_1)]/H,$$

$$\alpha_{2,1} = -\gamma_{41}(2\gamma_{12}A_2 + 6\mu\gamma_{11}A_1)/H, \qquad \alpha_{2,2} = -\gamma_{11}\gamma_{41}(A_2 - 6\mu\gamma_{31})/H,$$

$$\alpha_{2,3} = \gamma_{41}(\gamma_{11}A_1 + 2\gamma_{12}\gamma_{31})/H; \qquad \alpha_{3,0} = -6\gamma_{22}(\gamma_{11}A_3 + 2\gamma_{12}\gamma_{41})/H,$$

$$\alpha_{3,1} = [12\gamma_{12}\gamma_{22}\gamma_{41} - \gamma_{21}(2\gamma_{12}A_4 + 6\mu\gamma_{11}A_3)/H,$$

$$\alpha_{3,2} = [6\gamma_{11}\gamma_{22}\gamma_{41} - \gamma_{11}\gamma_{21}(A_4 - 6\mu\gamma_{41})]/H, \qquad \alpha_{3,3} = \gamma_{21}(\gamma_{11}A_3 + 2\gamma_{12}\gamma_{41})/H;$$

$$\alpha_{4,0} = 6\gamma_{22}(\gamma_{11}A_1 + 2\gamma_{12}\gamma_{31})/H, \qquad \alpha_{4,1} = \gamma_{21}(2\gamma_{12}A_2 + 6\mu\gamma_{11}A_1)/H,$$

$$\alpha_{4,2} = \gamma_{11}\gamma_{21}(A_2 - 6\mu\gamma_{31})/H, \qquad \alpha_{4,3} = -\gamma_{21}(\gamma_{11}A_1 + 2\gamma_{12}\gamma_{31})/H,$$

where

$$H = 6\gamma_{41}\gamma_{22}[2\gamma_{12}\gamma_{31} + \gamma_{11}A_1] - \gamma_{21}[\gamma_{11}(A_1A_4 - A_2A_3) + 2\gamma_{12}(\gamma_{31}A_4 - \gamma_{41}A_2) + 6\mu\gamma_{11}(\gamma_{31}A_3 - \gamma_{41}A_1)],$$

$$A_1 = 2\left[\gamma_{31}\int_0^1 \frac{1}{b(\xi)}\,\mathrm{d}\xi + \gamma_{32}\right], \qquad A_2 = 6\left[\mu\gamma_{31} + \gamma_{31}\int_0^1 \frac{\xi}{b(\xi)}\,\mathrm{d}\xi + \gamma_{32}\right],$$

$$A_3 = 2\gamma_{41}\int_0^1\int_0^\xi \frac{1}{b(\zeta)}\,\mathrm{d}\zeta\,\mathrm{d}\xi,$$

$$A_4 = 6\mu\gamma_{41}\left[1 - \int_0^1 \frac{1}{s(\xi)}\,\mathrm{d}\xi\right] + 6\gamma_{41}\int_0^1\int_0^\xi \frac{\zeta}{b(\zeta)}\,\mathrm{d}\zeta\,\mathrm{d}\xi - 6\gamma_{42},$$

$$\gamma_{i1} = \beta_i/(1+\beta_i), \quad i = 1, 2, 3, 4, \qquad \gamma_{i2} = 1/(1+\beta_i). \quad i = 1, 2, 3, 4.$$

CASE 2: HINGED–HINGED

In this case, β_2 and β_4 are infinite, β_1 and β_3 are zero, the boundary excitations become

$$\bar{f}_1(\tau) = f_1^*(\tau), \qquad \bar{f}_2(\tau) = f_2(\tau), \qquad \bar{f}_3(\tau) = f_3^*(\tau), \qquad \bar{f}_4(\tau) = f_4(\tau).$$

The coefficients of the shifting functions are

$$\alpha_{1,0} = 0, \qquad \alpha_{1,1} = (3A_3 - A_4)/6, \qquad \alpha_{1,2} = -1/2, \qquad \alpha_{1,3} = 1/6;$$

$$\alpha_{2,0} = -1, \qquad \alpha_{2,1} = -1, \qquad \alpha_{2,2} = 0, \qquad \alpha_{2,3} = 0;$$

$$\alpha_{3,0} = 0, \qquad \alpha_{3,1} = -A_4/6, \qquad \alpha_{3,2} = 0, \qquad \alpha_{3,3} = 1/6;$$

$$\alpha_{4,0} = 0, \qquad \alpha_{4,1} = 1, \qquad \alpha_{4,2} = 0, \qquad \alpha_{4,3} = 0,$$

where

$$A_3 = 2\int_0^1\int_0^\xi \frac{1}{b(\zeta)}\,\mathrm{d}\zeta\,\mathrm{d}\xi, \qquad A_4 = 6\mu\left[1 - \int_0^1 \frac{1}{s(\xi)}\,\mathrm{d}\xi\right] + 6\int_0^1\int_0^\xi \frac{\zeta}{b(\zeta)}\,\mathrm{d}\zeta\,\mathrm{d}\xi.$$

CASE 3: CLAMPED–CLAMPED

In this case, β_1, β_2, β_3 and β_4 are infinite, the boundary excitations become

$$\bar{f}_1(\tau) = f_1(\tau), \qquad \bar{f}_2(\tau) = f_2(\tau), \qquad \bar{f}_3(\tau) = f_3(\tau), \qquad \bar{f}_4(\tau) = f_4(\tau).$$

The coefficients of the shifting functions are

$$\alpha_{1,0} = 0, \qquad \alpha_{1,1} = (A_2A_3 - A_1A_4)/H, \qquad \alpha_{1,2} = (A_4 - A_2)/H,$$

$$\alpha_{1,3} = (A_1 - A_3)/H;$$

$$\alpha_{2,0} = (A_2A_3 - A_1A_4) + 6\mu(A_3 - A_1), \qquad \alpha_{2,1} = -6\mu A_1/H,$$

$$\alpha_{2,2} = -(A_2 - 6\mu)/H, \qquad \alpha_{2,3} = A_1/H;$$

$$\alpha_{3,0} = 0, \qquad \alpha_{3,1} = -6\mu A_3/H, \qquad \alpha_{3,2} = -(A_4 - 6\mu)/H, \qquad \alpha_{3,3} = A_3/H;$$

$$\alpha_{4,0} = 0, \qquad \alpha_{4,1} = 6\mu A_1/H, \qquad \alpha_{4,2} = (A_2 - 6\mu)/H, \qquad \alpha_{4,3} = -A_1/H;$$

where

$$H = (A_2A_3 - A_1A_4) - 6\mu(A_3 - A_1), \qquad A_1 = 2\int_0^1 \frac{1}{b(\xi)}\,\mathrm{d}\xi,$$

$$A_2 = 6\left[\mu + \int_0^1 \frac{\xi}{b(\xi)}\,\mathrm{d}\xi\right],$$

$$A_3 = 2\int_0^1\int_0^\xi \frac{1}{b(\zeta)}\,\mathrm{d}\zeta\,\mathrm{d}\xi, \qquad A_4 = 6\mu\left[1 - \int_0^1 \frac{1}{s(\xi)}\,\mathrm{d}\xi\right] + 6\int_0^1\int_0^\xi \frac{\zeta}{b(\zeta)}\,\mathrm{d}\zeta\,\mathrm{d}\xi.$$

CASE 4: CLAMPED–HINGED

In this case, β_1, β_2 and β_4 are infinite, β_3 is zero, the boundary excitations become

$$\bar{f}_1(\tau) = f_1(\tau), \qquad \bar{f}_2(\tau) = f_2(\tau), \qquad \bar{f}_3(\tau) = f_3^*(\tau), \qquad \bar{f}_4(\tau) = f_4(\tau).$$

The coefficients of the shifting functions are

$$\alpha_{1,0} = 0, \qquad \alpha_{1,1} = (6A_3 - 2A_4)/H, \qquad \alpha_{1,2} = -6/H, \qquad \alpha_{1,3} = 2/H;$$

$$\alpha_{2,0} = (6A_3 - 2A_4) - 12\mu, \qquad \alpha_{2,1} = -12\mu/H, \qquad \alpha_{2,2} = -6/H, \qquad \alpha_{2,3} = 2/H;$$

$$\alpha_{3,0} = 0, \qquad \alpha_{3,1} = -6\mu A_3/H, \qquad \alpha_{3,2} = -(A_4 - 6\mu)/H, \qquad \alpha_{3,3} = A_3/H;$$

$$\alpha_{4,0} = 0, \qquad \alpha_{4,1} = 12\mu/H, \qquad \alpha_{4,2} = 6/H, \qquad \alpha_{4,3} = -2/H,$$

where

$$H = (6A_3 - 2A_4) + 12\mu, \qquad A_3 = 2\int_0^1\int_0^\xi \frac{1}{b(\zeta)}\,\mathrm{d}\zeta\,\mathrm{d}\xi,$$

$$A_4 = 6\mu\left[1 - \int_0^1 \frac{1}{s(\xi)}\,\mathrm{d}\xi\right] + 6\int_0^1\int_0^\xi \frac{\zeta}{b(\zeta)}\,\mathrm{d}\zeta\,\mathrm{d}\xi.$$

CASE 5: CLAMPED-FREE

In this case, β_1 and β_2 are infinite, β_3 and β_4 are zero, the boundary excitations become

$$\bar{f}_1(\tau) = f_1(\tau), \qquad \bar{f}_2(\tau) = f_2(\tau), \qquad \bar{f}_3(\tau) = f_3^*(\tau), \qquad \bar{f}_4(\tau) = f_4^*(\tau).$$

The coefficients of the shifting functions are

$$\alpha_{1,0} = 0, \qquad \alpha_{1,1} = 1, \qquad \alpha_{1,2} = 0, \qquad \alpha_{1,3} = 0;$$

$$\alpha_{2,0} = 1, \qquad \alpha_{2,1} = 0, \qquad \alpha_{2,2} = 0, \qquad \alpha_{2,3} = 0;$$

$$\alpha_{3,0} = 0, \qquad \alpha_{3,1} = 0, \qquad \alpha_{3,2} = 1/2, \qquad \alpha_{3,3} = 0;$$

$$\alpha_{4,0} = 0, \qquad \alpha_{4,1} = \mu, \qquad \alpha_{4,2} = 1/2, \qquad \alpha_{4,3} = -1/6.$$